Quantum Mechanics II. Additional problems

Problem 1 : **Decay of the Λ^0 baryon**

The Λ^0 baryon is a particle of spin 1/2 and charge 0. This particle can decay into a proton p^+ and a π^- meson, a particle with spin 0 and charge $-e$. Suppose that the Λ^0 is initially at rest. Conservation of energy and momentum fixes the energies of the decay products p^+ and π^-, leaving undetermined only the direction of the decay.

To describe the final state, we can thus call θ the angle between the momentum $\mathbf{P}_{p^+}$ of the proton and the z axis, and φ the corresponding azimuthal angle. The final state is entirely described by the two angles θ, φ (see Fig. 1).

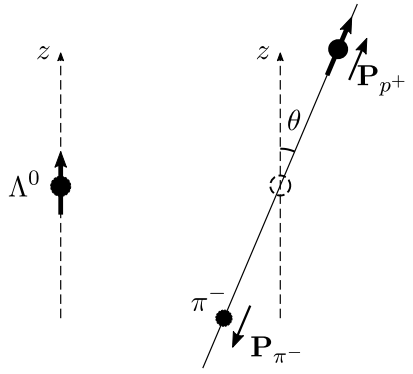

FIGURE 1 –

1. Suppose that the initial Λ^0 particle has spin $+1/2$ along the $+z$ axis and that after the decay the proton travels along the $+z$ direction (that is, the decay angle is $\theta = 0$). Show that the proton must have spin $+1/2$. In other words, if we prepare Λ^0 particles in the $+1/2$ state, and we place a detector equipped with a spin polarizer at large distance in the $\theta = 0$ direction, we will always observe protons with $S^z = +1/2$. Similarly, show that if the initial Λ^0 has spin $S = -1/2$ and the proton travels in the $+z$ direction then the proton spin must be $S^z = -1/2$.
2. Call $A_+ = \langle p^+, \pi^-; \theta = 0, S^z_{p^+} = +1/2 | U | \Lambda^0, S^z_{\Lambda^0} = +1/2 \rangle$ and $A_- = \langle p^+, \pi^-; \theta = 0, S^z_{p^+} = -1/2 | U | \Lambda^0, S^z_{\Lambda^0} = -1/2 \rangle$ the probability amplitudes that a Λ^0 baryon with spin $\pm 1/2$ in the z direction decays into a proton and a π^- meson with spin $\pm 1/2$, when the decay happens along the z axis. Here U is an unitary matrix and represents the time evolution $\mathrm{e}^{-iHt/\hbar}$ for a large time (sufficiently large so that the initial Λ^0 state has decayed into the final products).
   Calculate the total probability of decay in a direction θ, φ (summing over the two possible orientations of the proton spin), both in the case in which the Λ^0 particles has initially spin $S^z = +1/2$ and when it has initially spin $S^z = -1/2$.
   Hint. To study a decay happening in the (θ, φ) direction use the rotation matrices for spin 1/2 in order to analyze the spin of the initial Λ^0 particle in a basis in which the spin quantization axis is directed in the same direction of the decay direction. The decay amplitudes remain the same under a global rotation : if the initial spin, the final spin, and the momentum are all rotated simultaneously the amplitude of decay does not change. (For any rotation matrix, we must have $R^{-1} U R = R^+ U R = U$).
3. Assume that the initial state and the decay process conserve parity. The final distribution and the decay probability, then, must be symmetric under parity. Using the result of part 2. show that the only possibility is that $|A_+| = |A_-|$.

4. What is the probability distribution of decays if the initial state of the Λ^0 particle is a maximally mixed state, described by the density matrix $\rho = (|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)/2$?

Problem 2 : **Total number of states with a given total angular momentum**

Consider four particles of spin 1/2. Consider four particles of spin 1/2. How many states are possible in total ? What are the possible values of the quantum number J associated to the total angular momentum $\mathbf{J} = \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 + \mathbf{s}_4$? How many states there are for each of the possible values of J ?

Problem 3 : **Fine structure in an alkali atom**

The lowest energy states of an alkali atom can be analyzed in a first approximation by considering a single-particle Hamiltonian

$$H_0 = \frac{\mathbf{p}^2}{2m} + V(r) \ , \tag{1}$$

which describes the motion of the outer electron. Here $V(r) = V(|\mathbf{r}|)$ is an effective potential generated by the nucleus and by the inner electrons. Eq. (1) however is based on a non-relativistic description of the electron. The first relativistic corrections (in powers of v/c, where v is the velocity of the electron), introduce a "spin-orbit" interaction, coupling the coordinates to the spin of the electron. The spin-orbit interaction is described by a new term in the Hamiltonian [1]

$$\begin{aligned} H &= H_0 + H_{\rm so} \ , \\ H_{\rm so} &= \frac{\hbar}{2m^2c^2r}\frac{{\rm d}V(r)}{{\rm d}r}[\mathbf{s}\cdot(\mathbf{r}\times\mathbf{p})] = \frac{\hbar^2}{2m^2c^2r}\frac{{\rm d}V(r)}{{\rm d}r}(\mathbf{s}\cdot\mathbf{L}) \ . \end{aligned} \tag{2}$$

At zero order, when the effects of $H_{\rm so}$ are neglected, the spin and the orbital angular momentum $\mathbf{s}$, $\mathbf{L}$ are both conserved quantities : $[H_0, \mathbf{s}] = 0$, $[H_0, \mathbf{L}] = 0$. Therefore, ℓ, M, and s_z are good quantum numbers and the degeneracies are $2 \times (2\ell + 1)$. In the hydrogen atom, when $V(r) \propto 1/r$ there are additional degeneracies between states with different ℓ but the same n. This is no longer true for an arbitrary potential. The eigenstates of H_0 in general can be written in the basis $|n, \ell, m, s^z\rangle$. The energies are $H_0|n, \ell, m, s^z\rangle = E_0(n, \ell)|n, \ell, m, s^z\rangle$ and the degeneracy of the levels is $2 \times (2\ell + 1)$.

1. What is the symmetry group of the Hamiltonian H_0 and what is that of $H = H_0 + H_{\rm so}$. Do $\mathbf{s}$ and $\mathbf{L}$ commute with the full Hamiltonian H ? Does the total angular momentum $\mathbf{J} = \mathbf{s} + \mathbf{L}$ commute with H ? Discuss the relation between the symmetries and the conservation laws.

   Do you expect that the degeneracies of the levels of H_0 will be lifted after turning on the perturbation $H_{\rm so}$? What degeneracy do you expect in the levels of $H + H_{\rm so}$?
2. Write the Heisenberg equation of motion for the spin $\mathbf{s}$.
3. * Derive an expression for the corrections due to $H_{\rm so}$ in first order perturbation theory. Show that the splitting between the two "fine-structure" levels is $\Delta E = \hbar^2 j/(2m^2c^2)A_{n,\ell}$ where j is the total angular momentum quantum number and $A_{n,\ell} = \langle n, \ell, m, s^z|(1/r)\partial V/\partial r|n, \ell, m, s^z\rangle$ for any value of m, s^z (this constant does not depend on m and s^z).

Hints. The problem can be solved by combining first-order degenerate perturbation theory with group theory. Degenerate perturbation theory implies that the spectrum is determined by $E_0(n, \ell) +$

1. In addition to the spin-orbit coupling there are other relativistic corrections at the same order in v/c. These are neglected here.

ϵ_a, where ϵ_a are the eigenvalues of the $(4\ell+2)\times(4\ell+2)$ matrix $\langle n,\ell,m',s'^z|H_{\rm so}|n,\ell,m,s^z\rangle$. This matrix can be diagonalized, without any explicit computation, using eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}^z$. This leads to a problem of addition of angular momenta : in this case $\hat{\mathbf{J}}$ is the sum of the orbital momentum $\hat{\mathbf{L}}$ and the spin momentum $\hat{\mathbf{s}}$. To calculate the numerical value of the energy corrections it is useful to note that any quantum state which is an eigenstate of $\hat{\mathbf{L}}^2$, $\hat{\mathbf{s}}^2$, and $\hat{\mathbf{J}}^2$ with eigenvalues respectively $\ell(\ell+1)$, $s(s+1)$, and $j(j+1)$, is also an eigenstate of the operator $(\hat{\mathbf{s}}\cdot\hat{\mathbf{L}})=(\hat{\mathbf{J}}^2-\hat{\mathbf{L}}^2-\hat{\mathbf{s}}^2)/2$ with eigenvalue $(j(j+1)-\ell(\ell+1)-s(s+1))/2$.

Solution.

1. The non-perturbed Hamiltonian H_0 is invariant under rotations of the electron coordinates and under *independent* rotations of the spin $\mathbf{s}$. More precisely, the system has a symmetry group SO(3)×SU(2)× Z_2, since the Hamiltonian has also invariance under the parity transformation ($\mathbf{r}\to-\mathbf{r}$, $\mathbf{p}\to-\mathbf{p}$, $\mathbf{s}\to\mathbf{s}$) [2]. As a result, the energy eigenstates can be labeled as $|n,\ell,m,s_z\rangle$, where ℓ is the angular momentum, $m=L_z$, s_z is the z component of the spin, and n is an additional quantum number. The energy eigenvalues are $E_0(n,\ell)$: $H_0|n,\ell,m,S_z\rangle = E_0(n,\ell)|n,\ell,m,S_z\rangle$. Note that, in contrast with the hydrogen atom, the energy E_0 depends in general on *both* n and ℓ. In the hydrogen atom the fact that the energy depends only on n is due to the special form of the potential $V(r)\propto 1/r$. In general, however, $E_0(n,\ell)$ depends nontrivially on both the n and the ℓ quantum numbers.

   Consider now the Hamiltonian $H=H_0+H_{\rm so}$. H does not commute with $\mathbf{L}$ and $\mathbf{s}$; however it does commute with the total angular momentum $\mathbf{J}=\mathbf{L}+\mathbf{s}$. The total Hamiltonian is invariant under *simultaneous* rotations of the coordinates, the momentum, and the spin of the electron.

   Since the symmetry of H is lower than that of H_0 the degeneracy of the unperturbed levels will be lifted upon turning the perturbation. We can predict already without any explicit calculations that the levels will be characterized by the quantum number j corresponding to the total angular momentum $\hat{\mathbf{J}}^2=j(j+1)$, and that the degeneracies will be $2j+1$. The quantum number j identifies an irreducible representation of the group.

3. Within first-order perturbation theory we need to diagonalize the $2(2\ell+1)\times 2(2\ell+1)$ matrix $\langle n,\ell,m',s'_z|H_{\rm so}|n,\ell,m,s_z\rangle$. The eigenvectors of the matrix provide linear combinations of $|n,\ell,m,s_z\rangle$ which can be used as zero-order states in perturbation theory (if the degeneracy is already lifted at first-order in $H_{\rm so}$). However, since we know that $\hat{\mathbf{J}}$ is conserved we can solve the problem without having to compute any explicit diagonalization. In fact, since $\hat{\mathbf{J}}^2$ and $\hat{J}_z$ are conserved, j, and j_z are good quantum numbers which label the exact eigenstates of the Hamiltonian. In perturbation theory, this means that the linear combinations of $|n,\ell,m,s_z\rangle$ which provide the zero-order approximation to the exact wavefunctions, must also be states with a well defined value of $\hat{\mathbf{J}}^2$ and $\hat{j}_z$.

   The problem then reduces to an addition of angular momenta : we have to sum an angular momentum ℓ with an angular momentum $s=1/2$. The possible values of j are $\ell+1/2$ and $\ell-1/2$. As a result the level n,ℓ, which has initially degeneracy $4\ell+2$ in absence of the perturbation $H_{\rm so}$ becomes splitted into two levels, with degeneracies $(2j_1+1)=2\ell$ and $(2j_2+1)=2\ell+2$, respectively. These in atomic physics are called "fine-structure" level (in the case of an alkali atom with a single electron in the external shell).

   To calculate the energy in an explicit way, note that the eigenstates in leading-order perturbation theory must be, by the discussion above,

$$|n,\ell,j,j_z\rangle=\sum_{m=-\ell}^{\ell}\sum_{s^z=\pm1/2}|n,\ell,m,s^z\rangle\langle n,\ell,m,s^z|n,\ell,j,j_z\rangle\ , \tag{3}$$

2. In addition, one can also note that the system has time-reversal symmetry.

where the probability amplitudes $\langle n, \ell, m, s^z | n, j, j_z \rangle$ are given by the Clebsh-Gordan coefficient of the decomposition $\ell \otimes 1/2 = (\ell - 1/2) \oplus (\ell + 1/2)$. The energy of the two fine-structure levels can be calculated as

$$\begin{aligned} E_{\text{f.s.}}(j) &= \langle n, \ell, j, j_z | H_{\text{so}} | n, j, j_z \rangle \\ &= \frac{\hbar^2}{4m^2c^2} \left[j(j+1) - \ell(\ell+1) - \frac{3}{4} \right] \left\langle n, j, j_z \left| \frac{1}{r} \frac{\partial V}{\partial r} \right| n, j, j_z \right\rangle \ , \end{aligned} \tag{4}$$

where it was used that $s(s+1) = 3/4$ since the electron has spin $s = 1/2$.

The matrix element which remains to be calculated is of an operator which is scalar under rotations, and which commutes with both $\mathbf{s}$ and $\mathbf{L}$. In other words, it is scalar not only under simultaneous rotations of the coordinates and the spin, but also under independent rotations of them. To see the consequences of this, it is simpler to consider the matrix elements,

$$\left\langle n, \ell, m', s'^z \left| \frac{1}{r} \frac{\partial V}{\partial r} \right| n, \ell, m, s^z \right\rangle \tag{5}$$

in the initial basis. Since the operator does not change the spin the matrix element must be proportional to $\delta_{s^z s'^z}$. In addition, the matrix element is the same for $s^z = +1/2$ and for $s^z = -1/2$. We can thus reduce the calculation to a matrix element in the coordinate space (as for a spinless particle).

Since the operator $1/r(\partial V/\partial r)$ commutes with all generators $\mathbf{L}$ of the algebra and $|n, \ell, m\rangle$ form an irreducible representation of SO(3), the matrix elements $\langle n, \ell, m' | 1/r(\partial V/\partial r) | n, \ell, m \rangle$ must be proportional to the identity, that is $\langle n, \ell, m' | 1/r(\partial V/\partial r) | n, \ell, m \rangle = A_{n,\ell} \delta_{mm'}$.

So, overall, we have

$$\left\langle n, \ell, m', s'^z \left| \frac{1}{r} \frac{\partial V}{\partial r} \right| n, \ell, m, s^z \right\rangle = A_{n,\ell} \delta_{s^z s'^z} \delta_{mm'} \ . \tag{6}$$

This implies that, also in the basis of eigenstates,

$$\left\langle n, \ell, j', j'^z \left| \frac{1}{r} \frac{\partial V}{\partial r} \right| n, \ell, j, j^z \right\rangle = A_{n,\ell} \delta_{jj'} \delta_{j^z j'^z} \ , \tag{7}$$

is proportional to the identity.

Combining the results we see that the splitting between the two fine-structure levels is

$$\Delta E = \frac{j A_{n,\ell} \hbar^2}{2m^2c^2} \tag{8}$$

Remark. Since $\hat{\mathbf{L}}$ does not commute with H, $\hat{\mathbf{L}}^2$ is not an exact conserved quantity. Still, ℓ remains a good quantum number in perturbation theory at leading order, since by the general properties of degenerate perturbation theory the zero-order wavefunctions are superpositions of states within the same level (which have well defined values of $\hat{\mathbf{L}}^2$).

Problem 4 : **Bound state of two identical spin-1/2 particles**

Consider a composite particle composed by two identical spin-1/2 particles, bound together by an attractive potential. Suppose that the composite particle is at rest (the total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = 0$), and that it has total angular momentum $j = 3$ (here $\mathbf{j} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{s}_1 + \mathbf{s}_2$).

What is the total spin S and the total orbital angular momentum L of the two consituent spin-1/2 particles ?

Solution. Since the particle is at rest, its wavefunction cannot depend on the center-of-mass coordinate but only on the difference between the coordinates $\mathbf{r}_1$, $\mathbf{r}_2$ of the two particles. The wavefunction of the bound state, thus will have the form :

$$\psi(\mathbf{r}_1, s_1; \mathbf{r}_2, s_2) = \psi(\mathbf{r}_1 - \mathbf{r}_2; s_1, s_2) \ , \tag{9}$$

where $s_1 = \pm 1/2$, $s_2 = \pm 1/2$ are the projections of the spins along the z axis (or any other arbitrary quantization axis). To analyze the angular momentum of the state, we can expand $\varphi(\mathbf{r}_1 - \mathbf{r}_2; s_1, s_2)$ into spherical harmonics as

$$\psi(\mathbf{r}_1 - \mathbf{r}_2; s_1, s_2) = \sum_{\ell=0}^{\infty} \sum_{-m}^{m} Y_{\ell m}(\theta, \varphi) R_{n\ell}(|\mathbf{r}_1 - \mathbf{r}_2|; s_1, s_2) \ , \tag{10}$$

where θ and φ are spherical angles of the vector $\mathbf{r}_1 - \mathbf{r}_2$ in a spherical coordinate system. The total orbital angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ acts on the wavefunction as :

$$\begin{aligned} \mathbf{L}\psi(\mathbf{r}_1 - \mathbf{r}_2; s_1, s_2) &= (\mathbf{L}_1 + \mathbf{L}_2)\psi(\mathbf{r}_1 - \mathbf{r}_2; s_1, s_2) \\ &= (\mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2)\psi(\mathbf{r}_1 - \mathbf{r}_2; s_1, s_2) \\ &= (\mathbf{r}_1 \times \mathbf{p}_1 - \mathbf{r}_2 \times \mathbf{p}_1)\psi(\mathbf{r}_1 - \mathbf{r}_2) = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{p}_1 \psi(\mathbf{r}_1 - \mathbf{r}_2) \ . \end{aligned} \tag{11}$$

In the derivation, it was used that, since the wavefunction depends only on $\mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{p}_2\psi(\mathbf{r}_1 - \mathbf{r}_2) = -i\nabla_2\psi(\mathbf{r}_1 - \mathbf{r}_2) = +i\nabla_1\psi(\mathbf{r}_1 - \mathbf{r}_2) = -\mathbf{p}_1\psi(\mathbf{r}_1 - \mathbf{r}_2)$.

It follows that the state $Y_{\ell,m}(\theta, \varphi)R_{n\ell}(|\mathbf{r}_1 - \mathbf{r}_2|; s_1, s_2)$ has total angular momentum $L_z = m$ and $\hat{L}^2 = \ell(\ell + 1)$.

Since the particles have spin 1/2, the total spin can be either $S = 0$ or $S = 1$. From the addition of angular momenta we have then that the total angular momentum j can be equal to $\ell + 1$, ℓ, or $\ell - 1$, where ℓ is the orbital angular momentum. Since we know that $j = 3$, the possibilities are $(S = 0; \ell = 3; j = 3)$, $(S = 1; \ell = 2; j = 3)$, $(S = 1; \ell = 3; j = 3)$, $(S = 1; \ell = 4; j = 3)$ (or superpositions of them).

However, we also know that the two particles are identical and since $s = 1/2$, they have Fermi statistics. Thus, the wavefunction must be antisymmetric under exchange of the two particles.

The spherical harmonics $Y_{\ell,m}$ are symmetric for ℓ even and antisymmetric for ℓ odd. Since the singlet spin state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ is antisymmetric, the combination $(S = 0; \ell = 3)$ is forbidden for fermions.

The triplet spin states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ are symmetric under exchange, so from the other combinations, available, only $(S = 1; L = 3; J = 3)$ is allowed. To write down the wavefunctions explicitly, we need the Clebsh-Gordan coefficients for the $J = 3$ sector in the decomposition $1 \otimes 3 = 2 \oplus 3 \oplus 4$: the coefficients $\langle J = 3, J_z | S = 1, S_z; L = 3; L_z = M\rangle$.

Problem 5 : **Structure of the deuteron**

The nucleus of deuterium, called "deuteron" is a bound state of a neutron and a proton. The neutron has charge 0, spin 1/2, and mass m_n, the proton has charge $+e$, spin 1/2 and mass $m_p \approx m_n$. It is known that the deuteron has total spin $J = 1$ in the rest frame (the frame where the center of mass of the deuteron is at rest). (The total spin of the deuteron is $\mathbf{J} = \mathbf{L} + \mathbf{s}_1 + \mathbf{s}_2$, where $\mathbf{s}_1$ and $\mathbf{s}_2$ are, respectively, the spins of the proton and the neutron, and $\mathbf{L}$ is the total orbital angular momentum).

Questions.

1. Assume that the neutron and the proton interact via an attractive interaction which can be described by a central potential $V = V(|\mathbf{r}_1 - \mathbf{r}_2|)$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are, respectively, the coordinates of the proton and the neutron. The Hamiltonian is taken therefore in a first approximation as $H_0 = \mathbf{p}_1^2/(2m_p) + \mathbf{p}_2^2/(2m_n) + V(|\mathbf{r}_1 - \mathbf{r}_2|)$. From the fact that the spin of the deuteron is $J = 1$, deduce what are the possible values of the orbital angular momentum which can contribute to the deuteron wavefunction.

   Hint. Write all possible values of $\mathbf{L}$, $\mathbf{s}_1$, $\mathbf{s}_2$ and of their sum using the summation formulas for the angular momentum. Note that the total orbital angular momentum $\mathbf{L}$ is nonzero even if the center of mass of the deuteron is at rest, because of the relative motion between the proton and the neutron. (In the system in which the center of mass is at rest, $\mathbf{p}_2 = -\mathbf{p}_1$, so $\mathbf{L} = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{p}_1 = \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2) \times (\mathbf{p}_1 - \mathbf{p}_2)$).

2. Is the Hamiltonian H_0 sufficient to explain why the deuteron is observed always with $J = 1$? If not, what perturbation ΔH is needed in order to explain why the $J = 1$ states are selected ?

3. It can be assumed in a first approximation that the deuteron (which is the lowest energy bound state) is described by a state with $\ell = 0$ and $S = 1$. Assuming this approximation, analyze the response of the system to a weak external magnetic field. Show that the system has three Zeeman levels with energies $+\mu B$, 0, and $-\mu B$ and the magnetic moment is the sum of the intrinsic moments of the proton and the neutron $\mu = \mu_p + \mu_n$.

   Hint. If we turn on a small magnetic field $\mathbf{B}$, the Hamiltonian receives an additional term to first order in $\mathbf{B}$ equal to $-(\boldsymbol{\mu}_1 \cdot \mathbf{B} + \boldsymbol{\mu}_2 \cdot \mathbf{B} + e\hbar \mathbf{L} \cdot \mathbf{B}/(4m_p))$, where $\boldsymbol{\mu}_1 = 2\mu_p \mathbf{s}_1$, and $\boldsymbol{\mu}_2 = 2\mu_n \mathbf{s}_2$ are the intrinsic magnetic moments of the proton and the neutron. The last term, $\simeq e\hbar(\mathbf{L} \cdot \mathbf{B})/(4m_p) = \mu_N(\mathbf{L} \cdot \mathbf{B})/2$ arises from the fact that the proton is charged and thus its orbital motion inside the deuteron couples to the magnetic field. The constant $\mu_N = e\hbar/(2m_p)$ is called "nuclear magneton".

4. In reality, a better approximation to the deuteron state consists in a superposition of a dominant component with $\ell = 0$, $S = 1$, and a small component with $\ell = 2$, $S = 1$. For example the $|J^z = +1\rangle$ state can be written as a linear combination $\alpha|S = 1; \ell = 0; J = 1, J^z = 1\rangle + \beta|S = 1; \ell = 2; J = 1, J^z - 1\rangle$ where $|\alpha^2| + |\beta^2| = 1$ and $|S = 1; \ell = 0; J = 1, J^z = 1\rangle$, $|S = 1; \ell = 2; J = 1, J^z - 1\rangle$ are states with well defined values of S, ℓ, J, and J^z. Repeat the calculation of the Zeeman splitting for $\beta \neq 0$. Show that the energies are $+\mu$, 0, $-\mu$ but μ is now not equal to the sum $\mu_p + \mu_n$.

Solutions.

1. The total angular momentum of the deuteron is $\mathbf{J} = \mathbf{L} + \mathbf{s}_1 + \mathbf{s}_2$. From the summation formula for the angular momentum, the total spin $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$, can take two values, $S = 0$, $S = 1$. More precisely, the tensor product $\frac{1}{2} \otimes \frac{1}{2}$ of the two spin-1/2 Hilbert spaces breaks into a direct sum of a space with spin $S = 0$ and a space (of dimension 3) with spin 1 : $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. The state corresponding to $S = 0$, the "singlet", is $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. The three states forming the "triplet" of states with $S = 1$ are $|\uparrow\uparrow\rangle$, $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$, and $|\downarrow\downarrow\rangle$.

   The total momentum is the sum $\mathbf{L}+\mathbf{S}$ between the total spin and the orbital part. The sum can again be analyzed using the summation of angular momentum. The states in which the spins form a singlet, for which $\hat{\mathbf{S}}^2 = 0$, have simply $J = \ell$ and the angular momentum summation becomes simply $\ell \otimes 0 = \ell$. The states in which $S = 1$, instead, give three possible values of J : $J = \ell - 1$, $J = \ell$, and $J = \ell + 1$ (unless $\ell = 0$, in which case, the only value is $J = S = 1$).

   Knowing that the deuteron has $J = 1$ leaves the following possible values of the spin and the orbital momentum :

$$(S = 0, \ell = 1) \ , \qquad (S = 1, \ell = 0) \ , \qquad (S = 1, \ell = 1) \ , \qquad (S = 1, \ell = 2) \ . \tag{12}$$

2. The Hamiltonian $H_0 = \mathbf{p}_1^2/(2m) + \mathbf{p}_2^2/(2m) + V(|\mathbf{r}_1 - \mathbf{r}_2|)$ commutes with $\mathbf{L}$, $\mathbf{s}_1$, and $\mathbf{s}_2$. Since the two constituent particles are dinstinguishable, there is no constrain on the possible values of S and ℓ arising from the symmetry/antisymmetry of the wavefunction. As a result, if the Hamiltonian was exactly equal to H_0, we would expect that the bound states are four-time degenerate, with a well-defined value of ℓ, but not necessarily with a well defined value of S. Thus, explaining the fact that the lowest state has $J = 1$, requires some perturbation not present in H_0 which lifts the degeneracy. In particular, we need a perturbation which commutes with the total angular momentum $\mathbf{J}$, but which does *not* commute with $\mathbf{L}$, $\mathbf{s}_1$, and $\mathbf{s}_2$.

3. If we assume that the state has $\ell = 0$, the orbital contribution is identically zero. We thus have to calculate only the effect of $-(\boldsymbol{\mu}_1 \cdot \mathbf{B} + \boldsymbol{\mu}_2 \cdot \mathbf{B}) = -2(\mu_p \mathbf{s}_1 \cdot \mathbf{B} + \mu_n \mathbf{s}_2 \cdot \mathbf{B})$. In perturbation theory we need to diagonalize the perturbation only within the subspace corresponding to one energy level of the unperturbed Hamiltonian. In this case, we need to consider the matrix elements of $-2(\mu_p \mathbf{s}_1 \cdot \mathbf{B} + \mu_n \mathbf{s}_2 \cdot \mathbf{B})$ between states with $S = 1$.

   The simplest way to derive the spectrum is to assume that $\mathbf{B}$ is in the z direction. In this case, the stationary levels are $|\uparrow\uparrow\rangle$, $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle/\sqrt{2})$, and $|\downarrow\downarrow\rangle$ and the corresponding energies are $(\mu_p + \mu_n)$, 0, $-(\mu_p + \mu_n)$.

4. The Zeeman energy in a state of the type $\alpha|S = 1, \ell = 0; J = 1, J^z = +1\rangle + \beta|S = 1, \ell = 0; J = 1, J^z = +1\rangle$, to first order in perturbation theory, is given by the eigenvalues of the matrix

$$\begin{aligned} v_{M,M'} = (&\alpha^* \langle S = 1, \ell = 0; J = 1, J^z = M| \\ &+ \beta^* \langle S = 1, \ell = 2; J = 1, J^z = M|) \Delta H_B (\alpha|S = 1, \ell = 0; J = 1, J^z = M'\rangle \\ &+ \beta|S = 1, \ell = 2; J = 1, J^z = M'\rangle) \end{aligned} \tag{13}$$

   where $\Delta H_B = -\boldsymbol{\mu}_p \cdot \mathbf{B} - \boldsymbol{\mu}_n \cdot \mathbf{B} - \frac{1}{2}\mu_N \mathbf{L} \cdot \mathbf{B}$ is the perturbation induced by the magnetic field at first order and M, M' range over -1, 0, 1. Since the perturbation commutes with $\mathbf{L}^2$ there is no mixing of the $\ell = 0$ and the $\ell = 2$ components and we can simplify

$$\begin{aligned} v_{M,M'} = &|\alpha|^2 \langle S = 1, \ell = 0; J = 1, J^z = M|\Delta H_B|S = 1, \ell = 0; J = 1, J^z = M'\rangle \\ &+ |\beta|^2 (\langle S = 1, \ell = 2; J = 1, J^z = M|\Delta H_B|S = 1, \ell = 2; J = 1, J^z = M'\rangle) \ . \end{aligned} \tag{14}$$

   The term in the first line gives $-|\alpha|^2(\mu_p + \mu_n)(\mathbf{J} \cdot \mathbf{B})$. Thus if $|\alpha| = 1$ we recover the result of part 3., in which the magnetic moment μ is equal to the sum $\mu_p + \mu_n$. However, for $\beta \neq 0$ we also have a contribution from the second term.

   To analyze it, a possible way consists in assuming that the field is directed in the z direction. Since ΔH_B commutes with J^z the three states $M = 1$, $M = 0$, $M = -1$ are decoupled and the matrix $v_{M,M'}$ is diagonal. The contribution to the Zeeman splitting is then given by the diagonal elements

$$|\beta|^2 (\langle S = 1, \ell = 2; J = 1, J^z = M|\Delta H_B|S = 1, \ell = 2; J = 1, J^z = M\rangle) \tag{15}$$

   The state $|S = 1, \ell = 2; J = 1, J^z = M\rangle)$ can be expanded using the Clebsch-Gordan decomposition. From the tables of the Clebsch-Gordan series of the $1 \otimes 2 = 1 \oplus 2 \oplus 3$ we can read the

coefficients of the expansion,

$$
\begin{aligned}
|S=1;\ell=2;J=1;J^z=1\rangle &= \sqrt{\frac{3}{5}}|S=1;\ell=2;S^z=-1,\ell^z=2\rangle - \\
&\sqrt{\frac{3}{10}}|S=1;\ell=2;S^z=0,\ell^z=1\rangle + \sqrt{\frac{1}{10}}|S=1;\ell=2;S^z=1,\ell^z=0\rangle\ , \\
|S=1;\ell=2;J=1;J^z=0\rangle &= \sqrt{\frac{3}{10}}|S=1;\ell=2;S^z=-1,\ell^z=1\rangle - \\
&-\sqrt{\frac{2}{5}}|S=1;\ell=2;S^z=0,\ell^z=0\rangle + \sqrt{\frac{3}{10}}|S=1;\ell=2;S^z=1,\ell^z=-1\rangle\ , \\
|S=1;\ell=2;J=1;J^z=-1\rangle &= \sqrt{\frac{1}{10}}|S=1;\ell=2;S^z=-1,\ell^z=0\rangle - \\
&-\sqrt{\frac{3}{10}}|S=1;\ell=2;S^z=0,\ell^z=-1\rangle + \sqrt{\frac{3}{5}}|S=1;\ell=2;S^z=1,\ell^z=-2\rangle\ .
\end{aligned}
\tag{16}
$$

From these, we can calculate the averages

$$
\begin{aligned}
&\langle +1|S^z|+1\rangle = -\frac{3}{5}+\frac{1}{10} = -\frac{1}{2}\ , \quad \langle 0|S^z|0\rangle = -\frac{3}{10}+\frac{1}{10}=0\ , \\
&\langle -1|S^z|-1\rangle = -\frac{1}{10}+\frac{3}{5} = +\frac{1}{2}\ ,
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&\langle +1|\ell^z|+1\rangle = 2\times\frac{3}{5}+\frac{3}{10} = \frac{3}{2}\ , \quad \langle 0|\ell^z|0\rangle = -\frac{3}{10}+\frac{1}{10}=0\ , \\
&\langle -1|\ell^z|-1\rangle = -\frac{3}{10}-2\times\frac{3}{5} = -\frac{3}{2}\ ,
\end{aligned}
\tag{18}
$$

Here, to lighten the notation, the states $|S=1,\ell=2;J=1,J^z=M\rangle$ have been abbreviated as $|M\rangle$. All off-diagonal matrix elements are zero.

We are now almost ready to calculate the Zeeman splitting. We still need to compute the averages of the two individual spins s_1^z, s_2^z. This can be done using that the states which we are considering have $S=1$. For any matrix element inside this representation, $s_1^z = s_2^z = \frac{1}{2}S^z$. For example note that $\langle\uparrow\uparrow|s_1^z|\uparrow\uparrow\rangle = 1/2 = \frac{1}{2}\langle\uparrow\uparrow|S^z|\uparrow\uparrow\rangle$, and $[(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)/\sqrt{2}]s_1^z[(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} = 0 = \frac{1}{2}[(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)/\sqrt{2}]S^z[(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. Thus we have

$$
\begin{aligned}
&\langle M|s_1^z|M'\rangle = \langle M|s_2^z|M'\rangle = -\frac{1}{4}\langle M|J^z|M'\rangle = -\frac{1}{4}M\delta_{MM'}\ , \\
&\langle M|\ell^z|M'\rangle = \frac{3}{2}\langle M|J^z|M'\rangle = \frac{3}{2}M\delta_{MM'}\ .
\end{aligned}
\tag{19}
$$

Combining these with Eqs. (17) gives

$$
-B\langle M|(2\mu_p s_1^z + 2\mu_n s_2^z + \frac{1}{2}\mu_N\ell^z)|M\rangle = -BM\left(-\frac{1}{2}(\mu_p+\mu_n)+\frac{3}{4}\mu_N\right)\ . \tag{20}
$$

To get the final answer we need to sum the contributions from the $(\ell=0,S=1)$ part and from the $(\ell=2,S=1)$ part. The result is

$$
E_M = -M\mu B\ , \qquad \mu = |\alpha|^2(\mu_p+\mu_n) + \frac{|\beta|^2}{2}\left(\frac{3}{2}\mu_N - \mu_p - \mu_n\right)\ . \tag{21}
$$

The problem could also have been solved using the Wigner-Eckart theorem.